THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH4240 - Stochastic Processes - 2023/24 Term 2

Homework 6
Due: April 12 Friday (11:59 pm) 2024

Please submit online via Blackboard your answers to **all TEN questions below** including two supplementary questions. The late submission will not be accepted. Reference solutions will be provided after grading.

Exercises (Chapter 3, Page 107): 1, 2, 3, 4, 5, 6, 7, 10

Supplementary:

Q1. There are three states: 1 =sunny, 2 =smoggy, 3 =rainy. The weather stays sunny for an exponentially distributed number of days with mean 3, then becomes smoggy. It stays smoggy for an exponentially distributed number of days with mean 4, then rain comes. The rain lasts for an exponentially distributed number of days with mean 1, then sunshine returns. Let $\{X_t\}_{t\geq 0}$ be a Markov jump process to describe the weather. Find the rate matrix D and Markov matrix Q.

Q2. A factory has three machines in use and one repairman. Suppose each machine works for an exponential amount of time with mean 60 days between breakdowns, but each breakdown requires an exponential repair time with mean 4 days. Let $\{X_t\}_{t\geq 0}$ be a Markov jump process to describe the number of working machines. Find the rate matrix D and Markov matrix Q.

1. Solution. The forward equation is

$$\begin{cases} P'_{x0}(t) = -\lambda P_{x0}(t) + \mu P_{x1}(t), \\ P'_{x1}(t) = \lambda P_{x0}(t) - \mu P_{x1}(t), \end{cases} \qquad x = 0, 1, \ t \geq 0.$$

The rate matrix $D = \begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix}$. The eigenvalue of D are 0 and $-(\lambda + \mu)$ with their corresponding eigenvectors $(1, 1)^t$ and $(\lambda, -\mu)^t$ respectively.

Write $Q = \begin{pmatrix} 1 & \lambda \\ 1 & -\mu \end{pmatrix}$. Then

$$Q^{-1} D Q = \begin{pmatrix} 0 & 0 \\ 0 & -(\lambda + \mu) \end{pmatrix}.$$

Hence the transition functions are given by

$$\begin{aligned} P(t) = e^{Dt} &= Q \begin{pmatrix} 1 & 0 \\ 0 & e^{-(\lambda+\mu)t} \end{pmatrix} Q^{-1} \\ &= \begin{pmatrix} \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t} & \frac{\lambda}{\lambda+\mu} - \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t} \\ \frac{\mu}{\lambda+\mu} - \frac{\mu}{\lambda+\mu} e^{-(\lambda+\mu)t} & \frac{\lambda}{\lambda+\mu} + \frac{\mu}{\lambda+\mu} e^{-(\lambda+\mu)t} \end{pmatrix}, \quad t \geq 0. \end{aligned}$$

2. Solution. The rate matrix is given by

$$D = \begin{pmatrix} -\lambda_0 & \lambda_0 & 0 \\ \mu_1 & -(\lambda_1 + \mu_1) & \lambda_1 \\ 0 & \lambda_0 & -\lambda_0 \end{pmatrix}.$$

The forward equation is

$$\begin{cases} P'_{x0}(t) = -\lambda_0 P_{x0}(t) + \mu_1 P_{x1}(t), \\ P'_{x1}(t) = \lambda_0 P_{x0}(t) - (\lambda_1 + \mu_1) P_{x1}(t) + \lambda_0 P_{x2}(t), \\ P'_{x2}(t) = \lambda_1 P_{x1}(t) - \lambda_0 P_{x2}(t), \end{cases} \qquad x = 0, 1, 2, \ t \geq 0.$$

Note also that $P_{x0}(t) + P_{x1}(t) + P_{x2}(t) \equiv 1$, putting it into the second equation, we have

$$P_{x1}(t) = \lambda_0 - (\lambda_0 + \lambda_1 + \mu_1) P_{x1}(t), \quad x = 0, 1, 2, \ t \geq 0.$$

With initial condition $P_{01}(0) = 0$, we solve that

$$P_{01}(t) = \frac{\lambda_0}{\lambda_0 + \lambda_1 + \mu_1} - \frac{\lambda_0}{\lambda_0 + \lambda_1 + \mu_1} e^{-(\lambda_0 + \lambda_1 + \mu_1)t}.$$

Put this solution into the first equation, we have

$$P'_{00}(t) = -\lambda_0 P_{00}(t) + \frac{\mu_1 \lambda_0}{\lambda_0 + \lambda_1 + \mu_1} - \frac{\mu_1 \lambda_0}{\lambda_0 + \lambda_1 + \mu_1} e^{-(\lambda_0 + \lambda_1 + \mu_1)t}.$$

With the initial condition $P_{00}(0) = 1$,

$$P_{00}(t) = \frac{\mu_1}{\lambda_0 + \lambda_1 + \mu_1} + \frac{\lambda_1}{\lambda_1 + \mu_1} e^{-\lambda_0 t} + \frac{\lambda_0 \mu_1}{(\lambda_0 + \lambda_1 + \mu_1)(\lambda_1 + \mu_1)} e^{-(\lambda_0 + \lambda_1 + \mu_1)t}.$$

Finally,

$$\begin{aligned}
P_{02}(t) &= 1 - P_{00}(t) - P_{01}(t) \\
&= \frac{\lambda_1}{\lambda_0+\lambda_1+\mu_1} - \frac{\lambda_1}{\lambda_1+\mu_1}e^{-\lambda_0 t} + \frac{\lambda_0\lambda_1}{(\lambda_0+\lambda_1+\mu_1)(\lambda_1+\mu_1)}e^{-(\lambda_0+\lambda_1+\mu_1)t}.
\end{aligned}$$

3. Solution. Since the Poisson process $X(s)$ has the independent and stationary increments, we have

$$\begin{aligned}
P(X(s)=m|X(t)=n) &= \frac{P(X(s)=m, X(t)=n)}{P(X(t)=n)} \\
&= \frac{P(X(t)-X(s)=n-m)P(X(s)=m)}{P(X(t)=n)} \\
&= \frac{\frac{e^{-\lambda(t-s)}(\lambda(t-s))^{n-m}}{(n-m)!}\cdot\frac{e^{-\lambda s}(\lambda s)^m}{m!}}{\frac{e^{-\lambda t}(\lambda t)^n}{n!}} \\
&= \binom{n}{m}\left(\frac{s}{t}\right)^m\left(1-\frac{s}{t}\right)^{n-m} \sim B\left(n, \frac{s}{t}\right).
\end{aligned}$$

4. Solution. For $t > 0$, it is easy to see that two sets are equal: $\{T_m \le t\} = \{X(t) \ge m\}$. Hence,

$$\begin{aligned}
F_{T_m}(t) = P(T_m \le t) &= P(X(t) \ge m) \\
&= 1 - \sum_{k=0}^{m-1} P(X(t)=k) \\
&= 1 - \sum_{k=0}^{m-1} \frac{e^{-\lambda t}(\lambda t)^k}{k!}.
\end{aligned}$$

For $t \le 0$, as T_m is non-negative random variable so $\{T_m \le t\}$ is empty and hence $F_{T_m}(t) = 0$.

Remark: Many of you allowed $t = 0$ in $\dfrac{e^{-\lambda t}(\lambda t)^k}{k!}$, but this leads to $(\lambda t)^k = 0^0$ when $k = 0$, which is undefined.

5. Solution. Differentiate F_{T_m} to obtain the density of T_m. For $t \le 0$, we simply have $f_{T_m}(t) = 0$. For $t > 0$, we have

$$f_{T_m}(t) = e^{-\lambda t}\left(\sum_{k=1}^{m-1}\frac{\lambda^k t^{k-1}}{(k-1)!} - \lambda\sum_{k=0}^{m-1}\frac{\lambda^k t^k}{k!}\right) = \frac{\lambda^m t^{m-1} e^{-\lambda t}}{(m-1)!}.$$

(It is of Gamma distribution of parameter m and λ.)

6. Solution. We calculate directly that

$$\begin{aligned}
P(T_1 \le s | X(t) = n) &= P(X(s) \ge 1 | X(t) = n) \\
&= 1 - P(X(s) = 0 | X(t) = n) \\
&= 1 - \binom{n}{0} \left(\frac{s}{t}\right)^0 \left(1 - \frac{s}{t}\right)^n \quad \text{(by Q3)} \\
&= 1 - \left(1 - \frac{s}{t}\right)^n.
\end{aligned}$$

7. Solution. Using the hints, for any nonnegative integer n, we have

$$\begin{aligned}
P(X(T) = n) &= \int_0^\infty f_T(t) P(X(T) = n | T = t) dt \\
&= \int_0^\infty f_T(t) P(X(t) = n) dt \\
&= \int_0^\infty v e^{-vt} \frac{(\lambda t)^n e^{-\lambda t}}{n!} dt \\
&= \frac{v\lambda^n}{n!} \int_0^\infty t^n e^{-(v+\lambda)t} dt \\
&= \frac{v\lambda^n}{(v+\lambda)^{n+1}}.
\end{aligned}$$

The last step follows from successive integration by parts.

10. Solution. (a) The forward equation is

$$\begin{cases} P'_{xy}(t) = -\mu_y P_{xy}(t) + \mu_{y+1} P_{x,y+1}(t), & y \le x - 1, \\ P'_{xx}(t) = -\mu_x P_{xx}(t), & y = x. \end{cases}$$

(b) Directly solve the second equation with initial condition $P_{xx}(0) = 1$,

$$P_{xx}(t) = e^{-\mu_x t}.$$

(c) For $x = y$, it is done in (b). For $x < y$, $P_{xy}(t) = 0$. Now only consider the case $x > y$. Multiplying the integrating factor $e^{\mu_y t}$ on both sides in the first equation, we obtain

$$(e^{\mu_y t} P_{xy}(t))' = \mu_{y+1} e^{\mu_y t} P_{x,y+1}(t).$$

Integrating both side, and note that $P_{xy}(t) = 0$ for all $x > y$, we have

$$P_{xy}(t) = \mu_{y+1} \int_0^t e^{-\mu_y (t-s)} P_{x,y+1}(s) ds.$$

(d) Put $y = x - 1$ and the solution in (b) into the equation in (c), and then integrate directly,

$$P_{x,x-1}(t) = \begin{cases} \frac{\mu_x}{\mu_{x-1} - \mu_x} (e^{-\mu_x t} - e^{-\mu_{x-1} t}), & \mu_{x-1} \ne \mu_x, \\ \mu_x t e^{-\mu_x t}, & \mu_{x-1} = \mu_x. \end{cases}$$

(e) This is proved directly by backward mathematical induction on y from x to 0. It clearly holds by (b) when $y = x$. Assume it hols for $y+1$, then for y, by (c),

$$\begin{aligned}
P_{xy}(t) &= (y+1)\mu \int_0^t e^{-y\mu(t-s)} \binom{x}{y+1} (e^{-\mu s})^{y+1} (1-e^{-\mu s})^{x-y-1} ds \\
&= (y+1)\binom{x}{y+1} \mu e^{-y\mu t} \int_0^t e^{-\mu s}(1-e^{-\mu s})^{x-y-1} ds \\
&= (y+1)\binom{x}{y+1} (e^{-\mu t})^y \int_1^{e^{-\mu t}} (1-u)^{x-y-1} du \\
&= (y+1)\binom{x}{y+1} (e^{-\mu t})^y \frac{(1-e^{-\mu t})^{x-y}}{x-y} \\
&= \binom{x}{y} (e^{-\mu t})^y (1-e^{-\mu t})^{x-y}.
\end{aligned}$$

This completes the induction step.

SQ1. The rate matrix D is

$$D = \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} -1/3 & 1/3 & 0 \\ 0 & -1/4 & 1/4 \\ 1 & 0 & -1 \end{pmatrix} \end{array}$$

and the Markov matrix Q is

$$Q = \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \end{array}$$

SQ2. Let $X(t)$ be the number of machine in operations. Then $\mathcal{S} = \{0, 1, 2, 3\}$. For $x \in \{0, 1, 2\}$, as there is one repairman to repair a machine, we have $q_{x,x+1} = 1/4$. Denote the breakdown time of each machine to be Y_1, Y_2 and Y_3 respectively. Then, each $Y_i \sim Exp(1/60)$ for $i = 1, 2$ and 3. Note $\min\{Y_1, Y_2\} \sim Exp(1/60 + 1/60)$ and $\min\{Y_1, Y_2, Y_3\} \sim Exp(1/60 + 1/60 + 1/60)$. For $x \in \{1, 2, 3\}$, $q_{x,x-1} = 1/E(\min_{k=1,\dots x} Y_k) = x/60$ Therefore, we can write

The rate matrix D is

$$D = \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} -1/4 & 1/4 & 0 & 0 \\ 1/60 & -4/15 & 1/4 & 0 \\ 0 & 1/30 & -17/60 & 1/4 \\ 0 & 0 & 1/20 & -1/20 \end{pmatrix} \end{array}$$

and the Markov matrix Q is

$$
\begin{array}{cc}
 & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\
Q = & \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1/16 & 0 & 15/16 & 0 \\ 0 & 2/17 & 0 & 15/17 \\ 0 & 0 & 1 & 0 \end{pmatrix}
\end{array}
$$